Module - I

A partial differential equation (PDE) is an equation that contains partial derivatives of an unknown functions. In a PDE the dependent variable depends on two or more independent variable. Let Z be a variable which depends on the variables x & y then we have the following notation

$$p = \frac{\partial Z}{\partial x} = Z_x \qquad q = \frac{\partial Z}{\partial y} = Z_y$$

$$r = \frac{\partial^2 Z}{\partial x^2} = Z_{xx} \qquad s = \frac{\partial^2 Z}{\partial x \partial y} = Z_{yx}$$

$$t = \frac{\partial^2 Z}{\partial y^2} = Z_{yy}$$

FORMATION OF A PDE BY ELIMINATING ARBITRARY CONSTANT

→ If the no: of arbitrary constants to be eliminated is equal to the no: of independent variables, then PDE formed is of I^{st} order.

→ If the no: of arbitrary constant to be eliminated is more than the no: of independ: variables, then the PDE formed is of ~~2nd~~ second or higher order

1. Find the PDE by eliminating arbitrary constant

$$z = (x+a)(y+b)$$

Indep: varia $\rightarrow 2$

Arbitrary const $\rightarrow 2$. form PDE of 1^{st} order.

$$z = (x+a)(y+b) \quad \text{--- (1)}$$

diff (1) w.r.t. x

$$\frac{\partial z}{\partial x} = y+b \implies p = y+b \quad \text{--- (2)}$$

diff (1) w.r.t. y

$$\frac{\partial z}{\partial y} = x+a \implies q = x+a \quad \text{--- (3)}$$

Substituting eqn (2) & (3) in (1)

we have. $\underline{\underline{z = pq}}$.

2. $z = ax + by + a^2 + b^2$

$$z = ax + by + a^2 + b^2 \quad \text{--- (1)}$$

diff (1) w.r.t. x

$$\frac{\partial z}{\partial x} = a \implies p = a \quad \text{--- (2)}$$

diff (1) w.r.t y

$$\frac{\partial z}{\partial y} = b \implies q = b \quad \text{--- (3)}$$

Subst: (2) & (3) in (1) we have

$$z = px + qy + p^2 + q^2$$

3. $z = (x^2+a)(y^2+b)$ — ①

$\frac{\partial z}{\partial x} = 2x(y^2+b) \Rightarrow y^2+b = \frac{p}{2x}$ — ②

$\frac{\partial z}{\partial y} = 2y(x^2+a) \Rightarrow x^2+a = \frac{q}{2y}$

$Z = \frac{q}{2y} \cdot \frac{p}{2x}$ or $Z = \frac{pq}{4xy}$

4. $Z = (x-a)^2 + (y-b)^2$

$\frac{\partial z}{\partial x} = 2(x-a) \Rightarrow p = 2(x-a)$ or

$x-a = \frac{p}{2}$

$\frac{\partial z}{\partial y} = 2(y-b) \Rightarrow q = 2(y-b)$ or

$y-b = \frac{q}{2}$

$Z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$

5. $2Z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$2p = \frac{2x}{a^2}$ or $\frac{1}{a^2} = \frac{p}{x}$ — ②

$2q = \frac{2y}{b^2}$ or $\frac{1}{b^2} = \frac{q}{y}$ — ③

$2Z = px + qy$

6. Find the differential equation of whole Spheres of fixed radius having their cent on XY plane

The equation of the sphere with centre $(a, b, 0)$ and radius r is given by

$$(x-a)^2 + (y-b)^2 + (z-0)^2 = r^2 \quad \text{--- (1)}$$

$a, b \longrightarrow$ 2 arbit: Const = no: of independent variables, So form 1st order PDE

Diff: (1) partially w.r.t. 'x', we have.

$$2(x-a) + 0 + 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$\Longrightarrow x - a = -z\frac{\partial z}{\partial x} = -zp. \quad \text{--- (2)}.$$

Diff (1) partially w.r.t 'y' we have.

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0.$$

$$\Longrightarrow y - b = -zq \quad \text{--- (3)}$$

Substituting (2) & (3) in (1) we have.

$$(-pz)^2 + (-qz)^2 + z^2 = r^2$$

$$p^2z^2 + q^2z^2 + z^2 = r^2$$

$$z^2(p^2+q^2+1) = r^2$$

U.Q.
7. Form the PDE by eliminating a, b, c from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Soln

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{--- (1)}$$

~~Diff~~ arb: const — 3 No: of indep: variable = 2,

higher order → 2nd order

Diff (1) partially w.r.t 'x'.

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0 \Longrightarrow \frac{2x}{a^2} = \frac{-2z}{c^2} p.$$

or $$\frac{c^2}{a^2} = \frac{-zp}{x} \quad \text{--- (2)}.$$

Diff (2) partially w.r.t 'x'

$$-\left[\frac{x\left(zr + p\cdot\frac{\partial z}{\partial x}\right) - zp}{x^2}\right] = 0$$

$$zp - zxr - p^2x = 0$$

U.Q.
8. Find the partial diff: eqn: representing the family of spheres whose centres lies on Z-axis.

FORMATION OF PDE BY ELIMINATING ARBITRARY FUNCTIONS

Let u & v be 2 functions of x, y & z which is connected by the relation $\phi(u,v)=0$ where ϕ is an arbitrary function, then the solution of this equation is of the form

$$\boxed{Pp + Qq = R}$$

where $P = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} = u_y v_z - u_z v_y$

$Q = \begin{vmatrix} u_z & u_x \\ v_z & v_x \end{vmatrix} = u_z v_x - u_x v_z$

$R = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x$

I. Obtain the PDE by eliminating the arbitrary function from the following

1. $\phi(x^2+y^2, yz - y^2) = 0$

$u = x^2 + y^2 \qquad v = yz - y^2$

$u_x = 2x \qquad v_x = 0 \qquad v_z = y$

$u_z = 0 \qquad u_y = 2y \qquad v_y = z - 2y$

$P = \cancel{U_y V_z}\ U_y V_z - U_z V_y = 2y^2$

$Q = U_z V_x - U_x V_z = -2xy$

$R = U_x V_y - U_y V_x = 2xz - 4xy$

$\therefore$ the PDE is $Pp + Qq = R$

ie $2y^2 p - 2xyq = 2xz - 4xy$

2. $\phi(x+y+z,\ x^2+y^2+z^2) = 0$

$u = x+y+z \qquad v = x^2+y^2+z^2$

$U_x = U_y = U_z = 1 \qquad V_x = 2x,\ V_y = 2y\ \ V_z = 2z$

$P = 2z - 2y \qquad Q = 2x - 2z \qquad R = 2y - 2x.$

the PDE $Pp + Qq = R$ is.

$2(z-y)p + 2(x-z)q = 2(y-x)$

$\Rightarrow (z-y)p + (x-z)q = y-x$

3. $f(x+y+z,\ xyz) = 0$

$u = x+y+z \qquad V = xyz$

$U_x = U_y = U_z = 1 \qquad V_x = yz,\ V_y = xz,\ V_z = xy$

$P = xy - xz = x(y-z) \qquad Q = yz - xy = y(z-x)$

$R = xz - yz = z(x-y)$

$\therefore$ the PDE is $Pp + Qq = R$

ie $x(y-z)p + y(z-x)q = z(x-y)$

4. U.Q

Obtain the partial differential equation by eliminating the arbitrary function from $z = f(x^2+y^2)$

Here only one arbit: func: so form PDE of I^{st} order

$$z = f(x^2+y^2) \quad \text{--- (1)}$$

Differentiating (1) partially w.r.t. x.

$$\frac{\partial z}{\partial x} = f'(x^2+y^2) \cdot 2x \Rightarrow \frac{p}{2x} = f'(x^2+y^2) \quad \text{--- (2)}$$

Diff: (1) partially w.r.t y

$$\frac{\partial z}{\partial y} = f'(x^2+y^2) \cdot 2y \Rightarrow \frac{q}{2y} = f'(x^2+y^2) \quad \text{--- (3)}$$

from (2) & (3) we have $\frac{p}{2x} = \frac{q}{2y}$

$$\Rightarrow py = qx \quad \text{or} \quad \boxed{py - qx = 0}$$

5. U.Q

Form the PDE by eliminating arbitrary functions from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ (form I^{st} order. 1 - arb: func)

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \quad \text{--- (1)}$$

Diff (1) partially w.r.t x.

$$\frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{-1}{x^2}\right)$$

$$f'\left(\frac{1}{x} + \log y\right) = \frac{-px^2}{2} \quad \text{--- (2)}$$

Diff (1) partially w.r.t y

$$\frac{\partial z}{\partial y} = 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right) + 2y$$

$$f'\left(\frac{1}{x} + \log y\right) = \cancel{\frac{yp}{2}} \quad \frac{(q-2y)}{2}y \quad \text{--- (3)}$$

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from ② & ③ we have

$$\frac{-px^2}{2} = \frac{qy - 2y^2}{2}$$

$$\Rightarrow px^2 + qy - 2y^2 = 0$$

U.Q

6. Obtain the PDE by eliminating f & g from

$z = x\, f(y) + y\, g(x)$ — ①   2 arb: func $\Rightarrow$ 2nd order.

diff ① partially w.r.t x.

$$p = \cancel{x}\, f(y) + y\, g'(x) \quad — ②$$

diff ① partially w.r.t y

$$q = x f'(y) + g(x) \quad — ③$$

diff ② part: w.r.t y

$$s = f'(y) + g'(x) \quad — ④$$

$$px + qy = x\, f(y) + xy\, g'(x) + yx\, f'(y) + y\, g(x)$$

$$= x\, f(y) + y\, g(x) + xy\,[g'(x) + f'(y)]$$

$$\boxed{px + qy = z + xys}$$

7. Obtain the PDE by eliminating the arbitrary function from $x^2+y^2+z^2 = f(xy)$

$u = x^2+y^2+z^2$ $v = xy$

$u_x = 2x,\ u_y = 2y,\ u_z = 2z$ $v_x = y,\ v_y = x,\ v_z = 0$

$P = -2xz$ $Q = 2yz$ $R = 2(x^2-y^2)$

$\therefore$ the PDE is $Pp + Qq = R$ is

$$-2xzp + 2yzq = 2(x^2-y^2)$$

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SOLUTIONS OF A PDE

The solution of a pde in some domain D of the independent variables x & y is a function that has all partial derivatives appearing in the eqn: & satisfying the equation everywhere in D

COMPLETE SOLUTION OR COMPLETE INTEGRAL

Any relation of the form $f(x, y, z, a, b) = 0$ which contains two arbitrary ~~functions~~ constants a & b and is a solution of a PDE of the form $g(x, y, z, p, q) = 0$ of first order is said to be complete solution or complete Integral where $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$

GENERAL SOLUTION OR GENERAL INTEGRAL

A relation of the form $f(u, v) = 0$ involving an arbitrary function f connecting two unknown functions u & v and satisfying a first order PDE of the form $g(x, y, z, p, q) = 0$ is called general solution or general Integral

PARTICULAR SOLUTION OR PARTICULAR INTEGRAL

The solution obtained by determining the arbitrary constant in the complete integral or the arbitrary function in the general integral by using the given condition is called particular solution or particular integral of that PDE

EQUATIONS SOLVABLE BY DIRECT INTEGRATION

Those equations which contains only one partial-derivatives can be solved by direct integration. However in the place of constants of integration, must use arbitrary function of variable kept as constants.

1. Solve $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$.

Integrating w.r.t x.

$$\frac{\partial u}{\partial t} = e^{-t} \sin x + f(t)$$

Integrating w.r.t 't'

$$u = -e^{-t} \sin x + h(t) + g(x)$$

2. Solve $\log\left(\dfrac{\partial^2 z}{\partial x \partial y}\right) = x + y$

$$\frac{\partial^2 z}{\partial x \partial y} = e^{x+y} = e^x \cdot e^y$$

Integr: w.r.t x. $\dfrac{\partial z}{\partial y} = e^y \cdot e^x + f(y)$

Integ: w.r.t y $z = e^x \cdot e^y + g(y) + u(x)$

3. Solve $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0$

Integ: w.r.t x

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{\cos(2x-y)}{2} = f(y)$$

Integrating w.r.t x

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{\sin(2x-y)}{4} = x f(y) + g(y)$$

Integrating w.r.t y

$$z + x^3y^3 - \frac{\cos(2x-y)}{4} = x h(y) + \phi(y) + u(x)$$

$$z = \frac{1}{4}\cos(2x-y) - x^3y^3 + xh(y) + \phi(y) + u(x)$$

4. Solve $\frac{\partial^2 z}{\partial x^2} = a^2 x$ given when $x=0$ $\frac{\partial z}{\partial x} = a\sin y$ & $\frac{\partial z}{\partial y} = 0$

Integr: w.r.t $x \Rightarrow \frac{\partial z}{\partial x} = \frac{a^2x^2}{2} + f(y)$ — ①

Integ: w.r.t 'x' $\Rightarrow z = \frac{a^2x^3}{6} + xf(y) + h(y)$ — ②

$x=0$ $\frac{\partial z}{\partial x} = a\sin y \Rightarrow a\sin y = f(y)$

$\therefore$ $z = \frac{a^2x^3}{6} + ax\sin y + h(y)$ — ③

$\frac{\partial z}{\partial \cancel{x}} = \cancel{\frac{3a^2x^2}{6}} + \cancel{a\sin y}$. Diffrentiating w.r.t y

③ $\Rightarrow \frac{\partial z}{\partial y} = ax\cos y + h'(y)$ — ④

when $x=0$, $\frac{\partial z}{\partial y} = 0 \Rightarrow 0 = 0 + h'(y) \Rightarrow h(y) = c$

5. Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, given that $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$ & $z=0$ when y is an odd multiple of $\pi/2$

Integrating w.r.t x,

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \quad \text{---} (1)$$

when $x = 0$, $\frac{\partial z}{\partial y} = -2\sin y$

$$-2\sin y = -\sin y + f(y) \Rightarrow f(y) = -\sin y$$

(1) becomes $\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$ --- (2)

Integrating (2) w.r.t y

$$z = \cos x \cos y + \cos y + g(x) \quad \text{---} (3)$$

when y is an odd multiple of $\pi/2$, $z = 0$

$$0 = 0 + 0 + g(x) \Rightarrow g(x) = 0$$

$\therefore$ (3) becomes $z = \cos x \cos y + \cos y$

$\boxed{z = \cos y (1 + \cos x)}$ is the solution.

LINEAR EQUATIONS OF FIRST ORDER

The linear 1st order PDE of the form $Pp+Qq=R$ where P, Q, R are functions of x, y, z only & $p=\frac{\partial z}{\partial x}$ & $q=\frac{\partial z}{\partial y}$ is called Lagrange's linear equation

Method for solving Lagrange's linear equation

1. Form the equation $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, This is known as Lagrange's auxiliary equation
2. By the method of grouping or by the method of multipliers or both solve the auxiliary equations to get two independent solutions $u(x,y,z)=c_1$ & $v(x,y,z)=c_2$

Method of grouping

→ Form the auxiliary equation

→ Take any pair say $\frac{dx}{P}=\frac{dy}{Q}$. where z is absent in this case

→ Solve it by any convinient method of solving ODE and obtain a solution of the form $u(x,y,z)=c_1$

→ The 2nd solution is obtained by setting another pair. If possible we can use the 1st solution to obtain the second solution.

Method of multiplier

If l, m, n are 3 multipliers, then by a well known principles of algebra, each fraction $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ equal to $\frac{l\,dx+m\,dy+n\,dz}{lP+mQ+nR}$. Choose l, m, n such

$lP + mQ + nR = 0$ then $l\,dx + m\,dy + n\,dz = 0$. Integrating we get $u(x,y,z) = c_1$. This method may be repeated to get another independent solution $v(x,y,z) = c_2$

This multipliers l, m, n are called Lagrangian multipliers.

U.Q.

1\. Solve $xp + yq = 3z$

This is in the form $Pp + Qq = R$

$P = x \quad Q = y \quad R = 3z$

The auxiliary eqn: is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

ie $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$

Consider $\frac{dx}{x} = \frac{dy}{y}$ which is variable separable

$$\int \frac{dx}{x} = \int \frac{dy}{y} \Rightarrow \log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1 \Rightarrow \log\left(\frac{x}{y}\right) = \log c_1$$

$$\frac{x}{y} = c_1 \quad \text{—} (1)$$

Consider $\frac{dy}{y} = \frac{dz}{3z} \Rightarrow \int \frac{dy}{y} = \frac{1}{3}\int \frac{dz}{z}$

$$\Rightarrow \log y = \frac{1}{3}\log z + \log c_2 \Rightarrow \log y - \log z^{1/3} = \log c_2$$

$$\log \frac{y}{z^{1/3}} = \log c_2 \Rightarrow \frac{y}{z^{1/3}} = c_2$$

$\therefore$ the solution is $\phi(u, v) = 0$

ie $\phi\left(\frac{x}{y}, \frac{y}{z^{1/3}}\right) = 0$

2. Solve $y^2zp + x^2zq = xy^2$

$P = y^2z \quad Q = x^2z \quad R = xy^2$

Aux: Eqn is $\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$

Consider. $\frac{dx}{y^2z} = \frac{dy}{x^2z} \Rightarrow \frac{dx}{y^2} = \frac{dy}{x^2} \Rightarrow x^2dx = y^2dy$

$\int x^2 dx = \int y^2 dy \Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + C_1$

$3C_1 = C_1$

$\frac{x^3 - y^3}{3} = C_1 \Rightarrow x^3 - y^3 = 3C_1 = C_1$ — 1

Consider $\frac{dx}{y^2z} = \frac{dz}{xy^2} \Rightarrow \int x\,dx = \int z\,dz$

$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + C_2 \Rightarrow \frac{x^2 - z^2}{2} = C_2 \Rightarrow x^2 - z^2 = 2C_2$. le

$= C_2$. — 2

$\therefore$ the solution is $\phi(u, v) = 0$

ie $\phi(x^3 - y^3, x^2 - z^2) = 0$

3. Solve $p - q = \log(x+y)$

This is in the form $Pp + Qq = R$

where $P = 1$ & $Q = -1$ $R = \log(x+y)$

$\therefore$ Aux: Eqn is $\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$

Consider $\frac{dx}{1} = \frac{dy}{-1} \Rightarrow x = -y + C_1 \Rightarrow x + y = C_1$ — 1

Consider $\frac{dx}{1} = \frac{dz}{\log(x+y)} \Rightarrow dx = \frac{dz}{\log C_1}$ from 1

$\log C_1\, dx = dz \Rightarrow x \log C_1 = z + C_2$

$x\log(x+y) - z = C_2$ — 2 $\therefore$ the solution is $\phi(x+y, x\log(x+y) - z) = 0$.

4. Solve $(z-y)p+(x-z)q = y-x$.

This is in the form $Pp+Qq=R$

$P = z-y \quad Q = x-z \quad R = y-x$.

Aux: Eqn is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

Choose Lagrange's multipliers ~~as~~ l, m, n as $1,1,1$ such that $lP+mQ+nR=0$

$1(z-y)+1(x-z)+1(y-x) = 0$

$\Rightarrow l\,dx+m\,dy+n\,dz=0 \Rightarrow dx+dy+dz=0$

$\Rightarrow \int dx+\int dy+\int dz = 0 \Rightarrow x+y+z = C_1$ — (1)

Choose Lagrange's multipliers as x, y, z

$lP+mQ+nR = x(z-y)+y(x-z)+z(y-x)$

$= 0 \Rightarrow x\,dx+y\,dy+z\,dz=0$

$\Rightarrow \int x\,dx+\int y\,dy+\int z\,dz = 0$

$\Rightarrow \dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2} = C_2$ or $x^2+y^2+z^2 = 2C_2 = C_2$ — (2)

$\therefore$ the solution is $\phi(u,v)=0$

ie $\phi(x+y+z,\ x^2+y^2+z^2) = 0$

5. Solve $x^2(y-z)p+y^2(z-x)q = z^2(x-y)$

$P = x^2(y-z) \quad Q = y^2(z-x) \quad R = z^2(x-y)$

Aux: Eqn is $\dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)}$

Choose Lagrange's multipliers as $\dfrac{1}{x^2}, \dfrac{1}{y^2}, \dfrac{1}{z^2}$

$lP+mQ+nR = \dfrac{1}{x^2}x^2(y-z)+\dfrac{1}{y^2}y^2(z-x)+\dfrac{1}{z^2}z^2(x-y)$

$= 0$

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$\Rightarrow ldx + mdy + ndz = 0 \Rightarrow \frac{1}{x^2}dx + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$

$\Rightarrow \int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = 0 \Rightarrow \frac{-1}{x} - \frac{1}{y} - \frac{1}{z} = C_1$

$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = C_1$ —— ①

Choose Lagrange's multipliers as $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

Such that $lP + mQ + nR = 0$

$\Rightarrow ldx + mdy + ndz = 0 \Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0 \Rightarrow \log x + \log y + \log z = \log C_2$

$\Rightarrow \log(xyz) = \log C_2 \Rightarrow xyz = C_2$ —— ②

U.Q

6. Solve $(mz - ny)p + (nx - lz)q = ly - mx$.

This is of the form $Pp + Qq = R$

$P = mz - ny \quad Q = nx - lz \quad R = ly - mx$.

Aux: Eqn is $\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$.

Choose Lagrange's multipliers as x, y, z.

Such that $lP + mQ + nR = 0$

$\Rightarrow ldx + mdy + ndz = 0 \Rightarrow xdx + ydy + zdz = 0$

$\Rightarrow \int x\,dx + \int y\,dy + \int z\,dz = 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_1$

$\Rightarrow x^2 + y^2 + z^2 = 2C_1 = C_1$ —— ①

Choose Lagrange's multipliers as l, m, n

Such that $lP + mQ + nR = l(mz - ny) + m(nx - lz) + n(ly - mx) = 0$

$\Rightarrow ldx + mdy + ndz = 0 \Rightarrow \int ldx + \int mdy + \int ndz = 0$

$\Rightarrow lx + my + nz = C_2$ —— ②

$\therefore$ the solution is $\phi(x^2 + y^2 + z^2, lx + my + nz) = 0$

7. U.Q Solve $xp - yq = y^2 - x^2$

This is of the form $Pp + Qq = R$

$P = x \quad Q = -y \quad R = y^2 - x^2.$

Auxiliary eqn is $\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2-x^2}$

Consider $\frac{dx}{x} = \frac{dy}{-y}$, it is variable separable.

$\int \frac{dx}{x} = \int \frac{dy}{-y} \Rightarrow \log x = -\log y + \log c_1$

$\Rightarrow \log x + \log y = \log c_1 \Rightarrow \log(xy) = \log c_1 \Rightarrow xy = c_1$ — (1)

Choose lagrange's multipliers as $x, y, 1$

$lP + mQ + nR = 0 \Rightarrow l\,dx + m\,dy + n\,dz = 0$

$\Rightarrow \int x\,dx + \int y\,dy + \int 1\,dz = 0.$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + z = c_2.$ — (2)

$\therefore$ the solution is $\phi\left(xy, \frac{x^2}{2} + \frac{y^2}{2} + z\right) = 0.$

8. U.Q Find the general solution of $(y^2+z^2)p - xyq - xz = 0$

$(y^2+z^2)p - xyq = xz$ is of the form $Pp + Qq = R$

$P = y^2 + z^2 \quad Q = -xy \quad R = xz$

Aux: Eqn $\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{xz}$

Consider $\frac{dy}{-xy} = \frac{dz}{xz} \Rightarrow \int \frac{dy}{y} = \int -\frac{dz}{z}$

$\log y = -\log z + \log c_1 \Rightarrow \log(yz) = \log c_1$

$\Rightarrow \log(yz) = c_1$

$\Rightarrow yz = c_1$ — (1)

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Choose the Lagrange's multipliers as $x, y, -z$

$lP + mQ + nR = 0 \Rightarrow l\,dx + m\,dy + n\,dz = 0$

$\Rightarrow x\,dx + y\,dy - z\,dz = 0$

$\Rightarrow \int x\,dx + \int y\,dy - \int z\,dz = 0$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = c$

$\Rightarrow x^2 + y^2 - z^2 = 2c = c_2$ — (2)

$\therefore$ The solution is $\phi(yz, x^2+y^2-z^2) = 0$

9. Solve $(x+y)zp + (x-y)zq = x^2+y^2$

$P = (x+y)z \quad Q = (x-y)z \quad R = x^2+y^2$

$$\frac{dx}{(x+y)z} = \frac{dy}{(x-y)z} = \frac{dz}{x^2+y^2}$$

Choose Lagrange multipliers as $x, -y, -z$

$lP + mQ + nR = 0 \Rightarrow x\,dx - y\,dy - z\,dz = 0$

$\Rightarrow \int x\,dx - \int y\,dy - \int z\,dz = 0$

$\Rightarrow \frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = c \Rightarrow x^2 - y^2 - z^2 = 2c$

$\Rightarrow x^2 - y^2 - z^2 = c_1$ — (1)

Consider $\frac{dx}{(x+y)z} = \frac{dy}{(x-y)z}$

$(x-y)\,dx = (x+y)\,dy$

$x\,dx - y\,dx = x\,dy + y\,dy$

$x\,dx - y\,dy = x\,dy + y\,dx \Rightarrow \frac{x^2}{2} - \frac{y^2}{2} = \int d(xy)$

$\Rightarrow \frac{x^2-y^2}{2} - xy = c_2$ — (2)

The solution is $\phi\left(x^2-y^2-z^2, \frac{x^2-y^2}{2} - xy\right) = 0$

U.Q

10. Solve $y^2 p - xyq = x(z-2y)$

$P = y^2 \quad Q = -xy \quad R = x(z-2y)$

Aux: eqn is $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

Consider $\frac{dx}{y^2} = \frac{dy}{-xy} \Rightarrow x\,dx = -y\,dy$

$\Rightarrow \int x\,dx = -\int y\,dy \Rightarrow \frac{x^2}{2} = \frac{-y^2}{2} + c$

$\Rightarrow x^2 + y^2 = 2c = C_1$ — ①

Consider $\frac{dy}{-xy} = \frac{dz}{x(z-2y)} \Rightarrow 2y\,dy - z\,dy = y\,dz$

$2y\,dy = y\,dz + z\,dy \Rightarrow 2y\,dy = d(zy)$

$\int d(zy) = \int 2y\,dy \Rightarrow zy = y^2 + C_2$

$\Rightarrow zy - y^2 = C_2$ — ②

$\therefore$ the solution is $\phi(x^2+y^2, zy-y^2) = 0$.

U.Q

11. Solve $p - 2q = 3x^2 \sin(y+2x)$

Aux: Eqn $\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y+2x)}$

Consider $\frac{dx}{1} = \frac{dy}{-2} \Rightarrow \int 2\,dx = \int -dy \Rightarrow 2x + y = C_1$ — ①

Consider $\frac{dx}{1} = \frac{dz}{3x^2 \sin(y+2x)} \Rightarrow dx = \frac{dz}{3x^2 \sin C_1}$

$\Rightarrow 3x^2 \sin C_1\,dx = dz \Rightarrow x^3 \sin C_1 = z + C_2 \Rightarrow$

$x^3 \sin(2x+y) - z = C_2$ — ②

METHOD OF SEPARATION OF VARIABLES

The method of separation of variables is used to determine the solution of a PDE. In this method we assume that the solution of a given PDE is a product of two functions each depends only on one independent variables.

ie $U = X(x)\,Y(y)$ where X is a function of x alone + Y is a function of y alone.

1. Solve $U_x + U_y = 0$ using method of separation of variables.

Let $U = \cancel{XY}\ XY$ —(1) be the solution.

$U_x = YX'$ $\quad U_y = XY'$

Substituting in the given PDE we have

$$YX' + XY' = 0 \implies \frac{X'}{X} = \frac{-Y'}{Y}$$

Let $\frac{X'}{X} = k \implies \frac{1}{X}\frac{dX}{dx} = k$.

$\implies \frac{dX}{X} = k\,dx$ Now integrating

$$\implies \int \frac{dX}{X} = k\int dx \implies \log X = kx + C_1$$

$$\implies X = e^{kx+C_1} \implies X = e^{kx}\cdot e^{C_1} \quad e^{C_1} = C_1$$

$$\implies X = e^{kx} C_1 \quad \text{—(1)}$$

Let $\frac{-Y'}{Y} = k$

$$\frac{-1}{Y}\frac{dY}{dy} = k \Rightarrow \frac{dY}{Y} = -k\,dy$$

$$\Rightarrow \int \frac{dY}{Y} = -k\int dy \Rightarrow \log Y = -ky + C_2$$

$$\Rightarrow Y = e^{-ky}.e^{C_2} \Rightarrow Y = e^{-ky}C_2 \text{ — (2)} \quad e^{C_2} = C_2$$

$\therefore$ the Solution is $U = XY$

$$U = C_1 e^{kx}.\,C_2 e^{-ky} = C_1 C_2 e^{k(x-y)}$$

$$\boxed{U = C_3\, e^{k(x-y)}} \qquad C_1 C_2 = C_3$$

2. Solve $x.\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$ using method of Separation of variables.

Let $U = XY$

$$\frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$$

$$xX'Y - 2yXY' = 0 \Rightarrow xX'Y = 2yXY'$$

$$\Rightarrow \frac{xX'}{X} = \frac{2yY'}{Y} \qquad \text{Let } \frac{X'x}{X} = k$$

$$\frac{x}{X}\frac{dX}{dx} = k \Rightarrow \int \frac{1}{X}dX = \int \frac{k}{x}dx.$$

$$\log X = k\log x + \log C_1 \Rightarrow \log X - \log x^k = \log C_1$$

$$\Rightarrow \log\left(\frac{X}{x^k}\right) = \log C_1 \Rightarrow \frac{X}{x^k} = C_1 \Rightarrow X = C_1 x^k \text{ — (1)}$$

Let $\frac{2yY'}{Y} = k \Rightarrow \frac{2y}{Y}.\frac{dY}{dy} = k$

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$$\int \frac{dY}{Y} = \int \frac{k}{2y}\, dy \Rightarrow \log Y = \frac{k}{2}\log y + \log c_2$$

$$\Rightarrow \log Y - \log y^{k/2} = \log c_2 \Rightarrow \log\left(\frac{Y}{y^{k/2}}\right) = \log c_2$$

$$\Rightarrow \frac{Y}{y^{k/2}} = c_2 \Rightarrow Y = c_2\, y^{k/2} \quad \text{—} (2)$$

$\therefore$ the solution is $U = XY = c_1 x^k \cdot c_2 y^{k/2}$

$$U = c_1 c_2\, x^k y^{k/2} = c_3\, x^k y^{k/2}$$

3. Using method of separation of variables solve

$U_{xy} - U = 0$

Let $U = XY$ — (1)

$U_x = YX'$ — (2) $U_{xy} = Y'X'$ — (3)

Substituting we have. $Y'X' - XY = 0 \Rightarrow Y'X' = XY$

$$\Rightarrow \frac{X'}{X} = \frac{Y}{Y'} = k$$

Let $\frac{X'}{X} = k \Rightarrow \frac{1}{X}\frac{dX}{dx} = k \Rightarrow \int \frac{1}{X}\, dX = \int k\, dx$

$$\Rightarrow \log X = kx + c_1 \Rightarrow X = e^{kx + c_1}$$

$e^{c_1} = c_1$

$$X = e^{kx} \cdot e^{c_1} = e^{kx} \cdot c_1 \quad \text{—} (4)$$

Let $\frac{Y}{Y'} = k \Rightarrow \cancel{X}\frac{dx}{dX} = k \quad Y \cdot \frac{dy}{dY} = k$

$$\frac{1}{k}\, dy = \frac{dY}{Y} \Rightarrow \int \frac{dY}{Y} = \int \frac{1}{k}\, dy$$

$$\Rightarrow \log Y = \frac{1}{k}y + c_2 \Rightarrow Y = e^{y/k} \cdot e^{c_2} = e^{y/k} \cdot c_2$$

$\therefore$ the Solution is $\cancel{y}$ $U = XY = e^{kx} c_1 \cdot e^{y/k} c_2$

$$\therefore U = c_1 c_2\, e^{kx + \frac{y}{k}} = c_3\, e^{kx + \frac{y}{k}}$$

4. Using method of separation of variables solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{where} \quad u(x,0) = 3e^{-5x}$$

Let $U = XT$ be the solution

$$\frac{\partial u}{\partial x} = TX' \qquad \frac{\partial u}{\partial t} = XT'$$

Substituting in the given eqn:

$$TX' = 2XT' + XT \Rightarrow TX' = X(2T' + T)$$

$$\frac{X'}{X} = \frac{2T' + T}{T}$$

Let $\frac{X'}{X} = k \Rightarrow \frac{1}{X}\frac{dX}{dx} = k \Rightarrow \frac{dX}{X} = k\,dx.$

$$\int \frac{dX}{X} = \int k\,dx \Rightarrow \log X = kx + c_1$$

$$\Rightarrow X = e^{kx + c_1} \Rightarrow X = C_1 e^{kx} \quad (1) \quad \boxed{e^{c_1} = C_1}$$

$$\frac{2T'}{T} + 1 = k \Rightarrow \frac{2T'}{T} = k-1 \Rightarrow \frac{1}{T}\frac{dT}{dt} = \frac{k-1}{2}$$

$$\Rightarrow \int \frac{1}{T}\,dT = \frac{k-1}{2}\int dt \Rightarrow \log T = \frac{(k-1)}{2}t + c_2$$

$$\Rightarrow T = e^{\frac{(k-1)}{2}t} \cdot e^{c_2} \Rightarrow T = C_2 e^{\frac{(k-1)}{2}t} \quad (2) \quad e^{c_2} = C_2$$

$$U = U(x,t) = C_1 C_2 e^{kx} \cdot e^{\frac{k-1}{2}t} = C_3 e^{kx + \frac{(k-1)}{2}t}$$

we have $u(x,0) = 3e^{-5x}$

$$3e^{-5x} = C_3 e^{kx} \Rightarrow C_3 = 3 \text{ \& } k = -5$$

$$\therefore u(x,t) = 3e^{-5x - 3t}$$

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5. Using method of separation of variables solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} - u \quad , \quad u(x,0) = 5e^{-3x}.$$

Let $U = XT$ be the solution

$$\frac{\partial u}{\partial x} = TX' \quad , \quad \frac{\partial u}{\partial t} = XT'$$

Substituting we have $TX' = 2XT' - XT$

$$\Rightarrow TX' = (2T' - T)X \Rightarrow \frac{X'}{X} = \frac{2T' - T}{T}$$

Let $\frac{X'}{X} = k \Rightarrow \frac{dX}{X} = k\,dx \Rightarrow X = c_1 e^{kx}$ — (1)

$$\frac{2T'}{T} - 1 = k \Rightarrow \frac{T'}{T} = \frac{k+1}{2} \Rightarrow T = c_2 e^{t\frac{(k+1)}{2}} \text{ — (2)}$$

$$u(x,t) = c_3 e^{kx + t\frac{(k+1)}{2}} \text{ — (3)}$$

we have $u(x,0) = 5e^{-3x}. \Rightarrow 5e^{-3x} = c_3 e^{+kx}.$

$$\Rightarrow c_3 = 5 \text{ \& } k = -3$$

$u(x,t) = 5e^{-3x-t}$. be the solution

6. Solve $\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} - u = 0$, $u(x,0) = 6e^{-3x}$

7. Solve $3u_x + 2u_y = 0$, $u(x,0) = 4e^{-x}$ by method of separation of variables

NON LINEAR PDE OF FIRST ORDER.

Those equations in which p and q occur than in the first degree are called non-linear partial differential equations of first order. The complete solution of such an equation contains only two arbitrary constants (ie, equal to the number of independent variables involved) and the P.I is obtained by giving particular values to the constants.

CHARPITS METHOD

It's a general method for finding the complete Integral of a non-linear PDE.

Consider the equation $f(x, y, z, p, q) = 0$ — ①

Since z depends on x & y we have.

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = pdx + qdy \quad — ②$$

Now if we can find another relation involving x, y, z, p, q such as $\phi(x, y, z, p, q) = 0$ — ③

then we can solve ① & ③ for p and q and substitute in ②. This will give the solution, provided ② is integrable. For this we have to form the Charpit's subsidiary equations. ie

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{\partial\phi}{0}.$$

An integral of these equations involving p or q or both can be taken

1. Solve $(p^2+q^2)y = qz$

Let $f(x,y,z,p,q) = (p^2+q^2)y - qz = 0$ —— ①

Charpit's subsidiary equations are

$$\frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}.$$

Consider $\dfrac{dp}{-pq} = \dfrac{dq}{p^2} \Rightarrow p\,dp + q\,dq = 0$

Integrating we have. $p^2+q^2 = c^2$ —— ②

To Solve ① & ② $\Longrightarrow$ put ② in ① $\Rightarrow q = \dfrac{c^2y}{z}$

using this we have $\cancel{p = c\sqrt{z^2 - c^2y^2}}$

$$p = c\sqrt{(z^2 - c^2y^2)}/z$$

Substituting this values we have.

$$dz = p\,dx + q\,dy = \frac{c}{z}\sqrt{z^2-c^2y^2}\,dx + \frac{c^2y}{z}\,dy$$

$$\Longrightarrow z\,dz - c^2y\,dy = c\sqrt{z^2-c^2y^2}\,dx.$$

$$\Longrightarrow \frac{\frac{1}{2}\,d(z^2-c^2y^2)}{\sqrt{z^2-c^2y^2}} = c\,dx$$ Integrating this

we have. $\sqrt{z^2-c^2y^2} = cx + a$

$\Longrightarrow z^2 = (a+cx)^2 + c^2y^2$ which is the required Complete Integral.

2. Solve $2z + p^2 + qy + 2y^2 = 0$

$$f(x, y, z, p, q) = 2z^2 + p^2 + qy + 2y^2 = 0 \quad \text{—} ①$$

Charpits subsidiary equations are.

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2+qy)} = \frac{dp}{2p} = \frac{dq}{4y+3q}$$

Consider $\frac{dx}{-2p} = \frac{dp}{2p} \Rightarrow dp = -dx$

$\Rightarrow p = -x + a.$ — ② Substituting this in ①

we have $q = \frac{1}{y}\left[-2z - 2y^2 - (a-x)^2\right]$

$\therefore dz = pdx + qdy = (a-x)dx - \frac{1}{y}\left[2z + 2y^2 + (a-x)^2\right]dy$

Multiplying both sides by $2y^2$.

we have. $\cancel{d}(2zy^2) = -\left[y^2(a-x)^2 + y^4\right] + b.$

$\Rightarrow y^2\left[(x-a)^2 + 2z + y^2\right] = b$ which is the desired Solution.

3. Solve $2xz - px^2 - 2qxy + pq = 0$

Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$

Charpits subsidiary equations are.

$$\frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dz}{px^2-2pq-2qxy} = \frac{dp}{2z-2qy} = \frac{dq}{0}$$

$dq = 0 \implies q = a.$

put $q = a$ in the given equation we get $p = \dfrac{2x(z-ay)}{x^2-a}.$

$\therefore \; dz = p\,dx + q\,dy$

$$\implies dz = \frac{2x(z-ay)}{x^2-a}\,dx + a\,dy$$

or $$\frac{dz - a\,dy}{z-ay} = \frac{2x}{x^2-a}\,dx$$

$$\implies \frac{d(z-ay)}{z-ay} = \frac{2x}{x^2-a}\,dx.$$

Now Integrating we have.

$$\log(z-ay) = \log(x^2-a) + \log b$$

$$z - ay = b(x^2-a)$$

or $$z = ay + b(x^2-a)$$